The extended Weyl group $\widetilde{W}(D_5^{(1)})$ as an extension of KNY's birational representation of $\widetilde{W}(A_1^{(1)} \times A_3^{(1)})$

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Abstract

We study the birational representation of $\widetilde{W}(A_1^{(1)} \times A_3^{(1)})$ proposed by Kajiwara-Noumi-Yamada (KNY) in the case of $m = 2$ and $n = 4$. It is shown that the equation can be lifted to an automorphism of a family of $A_3^{(1)}$ surfaces and therefore the group of Cremona isometries is $\widetilde{W}(D_5^{(1)})$ ($\supset \widetilde{W}(A_1^{(1)} \times A_3^{(1)})$). The equation can be decomposed into two mappings which are conjugate to the q-P_{VI} equation. It is also shown that the subgroup of Cremona isometries which commute with the original translation is isomorphic to $\mathbb{Z} \times \widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$.

1 Introduction

Since the singularity confinement criterion was introduced as a discrete analogue of the Painlevé test [2], many discrete analogues of Painlevé equations have been proposed and extensively studied [3, 10]. Discrete Painlevé equations have been considered as 2-dimensional non-autonomous birational dynamical systems which satisfy this criterion and which have limiting procedures to the (continuous) Painlevé equations. In recent years it was shown by Sakai that all of these (from the point of view of symmetries) are obtained by studying rational surfaces in connection with extended affine Weyl groups [11].

On the other hand, recently Kajiwara *et al* (KNY) [6] have proposed a birational representation of the extended Weyl groups $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ on the field of rational functions $\mathbb{C}(x_{ij})$, which is expected to provide higher order discrete Painlevé equations (however, this representation is not always faithful, for example it is not faithful in the case where m or n equals 1 and in the case of $m = n = 2$). In the case of $m = 2$ and $n = 3, 4$, the actions of the translations can be considered to be 2-dimensional non-autonomous discrete dynamical systems and therefore to correspond to discrete Painlevé equations. Special solutions and some properties of these equations have been studied by several authors [5, 8]. In the case of $m = 2$ and $n = 4$, the action of the translation was thought to be a symmetric form of the q-discrete analogue of Painlevé V equation (q-P_V). However, the symmetry $\widetilde{W}(A_1^{(1)} \times A_3^{(1)})$ does not coincides with any symmetry of discrete Painlevé equations in Sakai's list, (in the case of $m = 2$ and $n = 3$, it coincides with an equation, which is associated with a family of $A_3^{(1)}$ surfaces and whose symmetry

is $\widetilde{W}(A_1^{(1)} \times A_2^{(1)})$, in Sakai's list). So it is natural to suspect that the symmetry might be a subgroup of a larger group associated with some family of rational surfaces.

In this paper we show that in the case of $m = 2$ and $n = 4$ the action of the translation can be lifted to an automorphism of a family of rational surfaces of the type $A_3^{(1)}$, i.e. surfaces such that the type of the configuration of irreducible components of their anti-canonical divisors is $A_3^{(1)}$, and therefore that the group of these automorphisms is $\widetilde{W}(D_5^{(1)})$ (hence it is not q-P_V by Sakai's classification). The action can be decomposed into two mappings which are conjugate to the q-P_{VI} equation. It is also shown that the subgroup of automorphisms which commute with the original translation is isomorphic to $\mathbb{Z} \times \widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$.

2 Birational representation of $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$

The birational representation of $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ on $\mathbb{C}(x_{i,j})$ proposed by Kajiwara $et\ al$ (KNY) [6] is an action on $\mathbb{C}(x_{i,j})$ ($i = 0, 1, \dots, m-1$, $j = 0, 1, \dots, n-1$ and the indices i, j are considered in modulo $m\mathbb{Z}, n\mathbb{Z}$ respectively) defined as follows.

We write the elements of the Weyl group corresponding to the simple roots as

$$r_i \in W(A_{m-1}^{(1)}), \ \ s_j \in W(A_{n-1}^{(1)})$$

and the elements corresponding to the rotations of the Dynkin diagrams as

$$\pi \in \mathrm{Aut}(\mathrm{Dynkin}(A_{m-1}^{(1)})), \ \ \rho \in \mathrm{Aut}(\mathrm{Dynkin}(A_{n-1}^{(1)})).$$

The action of these elements on $\mathbb{C}(x_{i,j})$ are defined as

$$r_i(x_{ij}) = x_{i+1,j}\frac{P_{i,j-1}}{P_{ij}}, \quad r_i(x_{i+1,j}) = x_{ij}\frac{P_{ij}}{P_{i,j-1}}, \quad r_k(x_{ij}) = x_{ij}, \quad (k \neq i, i-1),$$

$$s_j(x_{ij}) = x_{i,j+1}\frac{Q_{i-1,j}}{Q_{ij}}, \quad s_j(x_{i,j+1}) = x_{ij}\frac{Q_{ij}}{Q_{i-1,j}}, \quad s_k(x_{ij}) = x_{ij}, \quad (k \neq j, j-1),$$

$$\pi(x_{ij}) = x_{i+1,j}, \quad \rho(x_{i,j}) = x_{i,j+1},$$

where

$$P_{ij} = \sum_{a=0}^{n-1}\left(\prod_{k=0}^{a-1} x_{i,j+k+1} \prod_{k=a+1}^{n-1} x_{i+1,j+k+1}\right), \quad Q_{ij} = \sum_{a=0}^{m-1}\left(\prod_{k=0}^{a-1} x_{i+k+1,j} \prod_{k=a+1}^{m-1} x_{i+k+1,j+1}\right).$$

For example in the $(m, n) = (2, 4)$ case,

$$P_{00} = x_{1,2}x_{1,3}x_{1,0} + x_{0,1}x_{1,3}x_{1,0} + x_{0,1}x_{0,2}x_{1,0} + x_{0,1}x_{0,2}x_{0,3}$$

and $Q_{00} = x_{0,1} + x_{1,0}$. It was shown by KNY that this action is a representation of $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ as automorphisms of the field $\mathbb{C}(x_{i,j})$. But it is still an open problem when this representation is faithful. In the case of $m = 2$ and $n = 3, 4$, one can see it is faithful by considering the actions on the root systems which we discuss later.

In the case of $(m, n) = (2, 4)$, the variable transformation:

$$\left(\frac{x_{0,j}x_{1,j}}{x_{0,j+1}x_{1,j+1}}\right)^{1/2} = a_j, \qquad \left(\frac{x_{0,j}x_{0,j+1}}{x_{1,j}x_{1,j+1}}\right)^{1/2} = f_j \ ,$$

reduces the actions of $r_0, r_1, s_0, s_1, s_2, s_3, \pi, \rho$ to

$$r_0(a_i) = a_i,$$

$$r_0(f_i) = \frac{1}{a_i a_{i+1} f_{i+1}} \frac{1 + a_i f_i + a_i a_{i+1} f_i f_{i+1} + a_i a_{i+1} a_{i+2} f_i f_{i+1} f_{i+2}}{1 + a_{i+2} f_{i+2} + a_{i+2} a_{i+3} f_{i+2} f_{i+3} + a_{i+2} a_{i+3} a_{i+4} f_{i+2} f_{i+3} f_{i+4}}$$

$$\pi(a_i) = a_i, \quad \pi(f_i) = \frac{1}{f_i}$$

$$r_1 = \pi \circ r_0 \circ \pi$$

$$s_i(a_j) = a_j a_i^{-c_{i,j}}, \quad s_i(f_j) = f_j \left(\frac{a_i + f_i}{1 + a_i f_i}\right)^{u_{i,j}}$$

$$\rho(a_i) = a_{i+1}, \quad \rho(f_i) = f_{i+1}$$

where $c_{i,j}$ and $u_{i,j}$ are

$$(c_{i,j}) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \quad (u_{i,j}) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Remark. By the variable transformation we have $a_0 a_1 a_2 a_3 = 1$, but if we remove this constraint and set $a_0 a_1 a_2 a_3 = q^{-1}$, the actions also generate $\widetilde{W}(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$.

The element $\pi \circ r_0$ is a translation of $A_1^{(1)}$ and provides a discrete dynamical system:

$$\bar{a}_0 = a_0, \quad \bar{a}_1 = a_1, \quad \bar{a}_2 = a_2, \quad \bar{a}_3 = a_3,$$

$$\bar{f}_0 = a_0 a_1 f_1 \frac{1 + a_2 f_2 + a_2 a_3 f_2 f_3 + a_2 a_3 a_0 f_2 f_3 f_0}{1 + a_0 f_0 + a_0 a_1 f_0 f_1 + a_0 a_1 a_2 f_0 f_1 f_2},$$

$$\bar{f}_1 = a_1 a_2 f_2 \frac{1 + a_3 f_3 + a_3 a_0 f_3 f_0 + a_3 a_0 a_1 f_3 f_0 f_1}{1 + a_1 f_1 + a_1 a_2 f_1 f_2 + a_1 a_2 a_3 f_1 f_2 f_3},$$

$$\bar{f}_2 = a_2 a_3 f_3 \frac{1 + a_0 f_0 + a_0 a_1 f_0 f_1 + a_0 a_1 a_2 f_0 f_1 f_2}{1 + a_2 f_2 + a_2 a_3 f_2 f_3 + a_2 a_3 a_0 f_2 f_3 f_0},$$

$$\bar{f}_3 = a_3 a_0 f_0 \frac{1 + a_1 f_1 + a_1 a_2 f_1 f_2 + a_1 a_2 a_3 f_1 f_2 f_3}{1 + a_3 f_3 + a_3 a_0 f_3 f_0 + a_3 a_0 a_1 f_3 f_0 f_1}.$$

Remark. Contrary to the case where these mappings are considered to be field operators, we define the composition of mappings as that of functions. For example, for $\varphi : (x, y) \mapsto (x^2, y)$ and $\psi : (x, y) \mapsto (x + y, y)$, we have $\psi \circ \varphi : (x, y) \mapsto (x^2 + y, y)$.

By the change of variables

$$a_3 = 1/(a_0 a_1 a_2 q), \quad f_0 = x, \ f_1 = y, \ f_2 = c/x, \ f_3 = d/y$$

this equation reduces to the following 2-dimensional non-autonomous discrete dynamical system:

$$\varphi : (x, y) \quad \mapsto \quad (\overline{x}, \overline{y}),$$

$$\begin{cases} \overline{x} = \dfrac{qa_0a_1xy + c(d + a_0dx + qa_0a_1a_2y)}{qx(1 + a_0(x + a_1a_2cy + a_1xy))} \\ \overline{y} = \dfrac{c(d + a_0dx + a_0a_1dxy + qa_0a_1a_2y)}{y(qa_0x(1 + a_1y) + c(d + qa_0a_1a_2y))}, \end{cases} \qquad (1)$$

where the change in the parameters is given by

$$(\overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{q}, \overline{c}, \overline{d}) = (a_0, a_1, a_2, q, \frac{d}{q}, \frac{c}{q}).$$

3 Space of initial conditions and Cremona isometries

3.1 Space of initial conditions

The notion of space of initial conditions (values) was first proposed by Okamoto [9] for the continuous Painlevé equations and was recently applied by Sakai [11] for the discrete Painlevé equations. In the discrete case it is linked to automorphisms of certain families of rational surfaces. The relations of surfaces and groups of these automorphisms were also studied by many authors from the algebraic-geometric view point [1, 7]. In this section, following Sakai's method, we construct the space of initial conditions for φ.

Let X and Y be rational surfaces and let X' and Y' be surfaces obtained by the successive blow-ups $\pi_X : X' \to X$ and $\pi_Y : Y' \to Y$. A rational mapping $\varphi' : X' \to Y'$ is called *lifted* from a rational mapping $\varphi : X \to Y$, if $\pi_Y \circ \varphi' = \varphi \circ \pi_X$ holds for any point where $\varphi \circ \pi_X$ and φ' are defined.

Definition(space of initial conditions). Let X_i's be rational surfaces and let $\{\varphi_i : X_i \to X_{i+1}\}$ be a sequence of birational mappings. A sequence of rational surfaces $\{Y_i\}$ is (or Y_i themselves are) called the space of initial conditions for $\{\varphi_i\}$ if each φ_i is lifted to an isomorphism, i.e. bi-holomorphic mapping, from Y_i to Y_{i+1}.

Let us consider the mapping (1) to be a birational mapping from $\mathbb{P}^1 \times \mathbb{P}^1$ to itself. Blowing up successively at the indeterminate points of φ and φ^{-1} (as usual), we have the surface X_i as described in Fig.1. The total transforms of the points of successive blow-ups are

$$\begin{array}{ll} E_1 : (x, y) = (0, -da_3) & E_2 : (x, y) = (0, -a_1) \\ E_3 : (x, y) = (-1/a_0, 0) & E_4 : (x, y) = (-c/a_2, 0) \\ E_5 : (1/x, y) = (0, -1/a_1) & E_6 : (1/x, y) = (0, -d/a_3) \\ E_7 : (x, 1/y) = (-ca_2, 0) & E_8 : (x, 1/y) = (-a_0, 0), \end{array} \qquad (2)$$

where we use $a_3 = 1/(a_0a_1a_2q)$ instead of q.

Theorem 3.1 *The mapping $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ (1) can be lifted to the sequence of isomorphisms $\{\varphi : X_i \to X_{i+1}\}$.*

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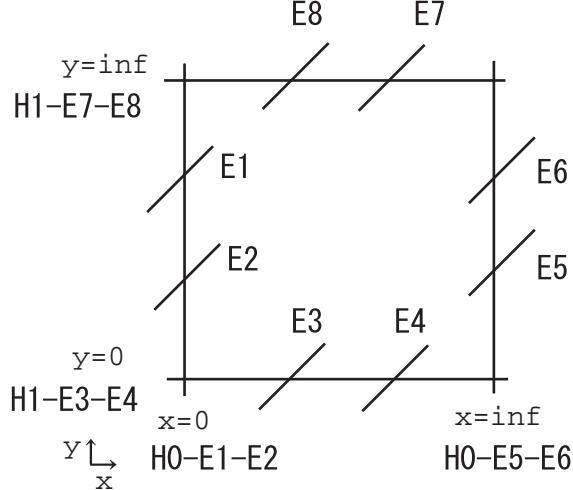
Figure 1: Irreducible divisors and their intersections

Let X be a rational surface which is given by blow-ups (2) from $\mathbb{P}^1 \times \mathbb{P}^1$, where $a_i, c, d \in \mathbb{C}\backslash\{0\}$. We denote the linear equivalence class of total transforms of $x = $ constant, $(y = $ constant$)$ on X by H_0 (resp. H_1). The Picard group of the surface X, i.e. the group of linear equivalence classes of divisors on X, is denoted as Pic(X). The mapping φ induces the push-forward mapping $\varphi_* : \mathrm{Pic(X_i)} \to \mathrm{Pic(X_{i+1})}$ $(\mathrm{Pic(X_i)} \simeq \mathrm{Pic(X_{i+1})})$ as

$$
\begin{aligned}
H_0 &\mapsto 2H_0 + H_1 - E_2 - E_4 - E_6 - E_8 \\
H_1 &\mapsto H_0 + 2H_1 - E_2 - E_4 - E_6 - E_8 \\
E_1 &\mapsto H_0 + H_1 - E_2 - E_4 - E_8 \qquad E_2 \mapsto E_5 \\
E_3 &\mapsto H_0 + H_1 - E_2 - E_4 - E_6 \qquad E_4 \mapsto E_7 \\
E_5 &\mapsto H_0 + H_1 - E_4 - E_6 - E_8 \qquad E_6 \mapsto E_1 \\
E_7 &\mapsto H_0 + H_1 - E_2 - E_6 - E_8 \qquad E_8 \mapsto E_3 \ .
\end{aligned}
$$

3.2   The group of Cremona isometries

In this section we shall consider automorphisms of the family of X with various parameters.

The Picard group of X is the $\mathbb{Z}$-module (lattice)

$$
\mathrm{Pic}(X) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_8.
$$

The intersection number of any two divisors can be calculated by using the intersection form

$$
H_i \cdot H_j = 1 - \delta_{i,j}, \ E_k \cdot E_l = -\delta_{k,l}, \ H_i \cdot E_k = 0 \ . \tag{3}
$$

Let D_0, D_1, D_2, D_3 be divisors as

$$
\begin{aligned}
D_0 &= H_0 - E_1 - E_2, \quad D_1 = H_1 - E_3 - E_4 \\
D_2 &= H_0 - E_5 - E_6, \quad D_3 = H_1 - E_7 - E_8.
\end{aligned}
$$

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The anti canonical divisor of X, $-K_X = 2H_0 + 2H_1 - E_1 - \cdots - E_8$, is uniquely decomposed into prime divisors (for generic parameters) as

$$-K_X = D_0 + D_1 + D_2 + D_3.$$

The connections of the D_i are expressed by the following diagram.

$$(4)$$

Such rational surfaces are said to be of the $A_3^{(1)}$ type which is the type of those surfaces in Sakai's list associated with the $q\text{-}P_{VI}$ equation .

Definition. An automorphism s of $\mathrm{Pic}(X)$ is called a *Cremona isometry* [1, 7] if s preserves i) the intersection form on $\mathrm{Pic}(X)$, ii) the canonical divisor K_X, iii) the semigroup of effective classes of divisors.

Let X and X' be rational surfaces. In general, if a mapping φ is an isomorphism from X to X', its action on the Picard group $\varphi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ (or $\varphi^* : \mathrm{Pic}(X') \to \mathrm{Pic}(X)$) $(\mathrm{Pic}(X) \simeq \mathrm{Pic}(X'))$ is always a Cremona isometry.

A Cremona isometry preserves the set $\{D_i\}$ and its orthogonal (with respect to the intersection form) lattice,

$$Q := \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4 \oplus \mathbb{Z}\alpha_5 \tag{5}$$

where the α_i's are

$$\begin{aligned}
\alpha_0 &= E_5 - E_6, & \alpha_1 &= E_1 - E_2 \\
\alpha_2 &= H_1 - E_1 - E_5, & \alpha_3 &= H_0 - E_3 - E_7 \\
\alpha_4 &= E_3 - E_4, & \alpha_5 &= E_7 - E_8 .
\end{aligned}$$

We define the action w_i on $\mathrm{Pic}(X)$ corresponding to α_i as

$$w_i(D) = D - \frac{2D \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i$$

where $D \in \mathrm{Pic}(X)$. The Cartan matrix is as follows.

$$(c_{i,j}) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & 2
\end{pmatrix}.$$

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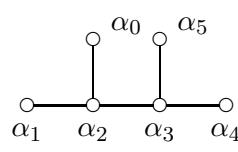
We also define the actions of the generators of automorphisms of the Dynkin diagrams σ_{102345} and σ_{543210} as

$$\sigma_{102345} : \quad (H_0, H_1, E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8)$$
$$\mapsto (H_0, H_1, E_5, E_6, E_3, E_4, E_1, E_2, E_7, E_8)$$
$$\sigma_{543210} : \quad (H_0, H_1, E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8)$$
$$\mapsto (H_1, H_0, E_3, E_4, E_1, E_2, E_7, E_8, E_5, E_6), \tag{6}$$

where the suffixes of σ mean corresponding permutations. For convenience we also use the automorphism $\sigma_{012354} = \sigma_{543210} \circ \sigma_{102345} \circ \sigma_{543210}$. For simplicity we write σ_{102345} and σ_{012354} as σ_{10} and σ_{54} respectively.

Here, the group of automorphisms of the lattice $\bigoplus \mathbb{Z}\alpha_i$ which preserve the intersection form (considered to be the inner product) is $\pm\widetilde{W}(D_5^{(1)})$ (see [4] §5.10). Since each element of $-\widetilde{W}(D_5^{(1)})$ does not preserve effectiveness of divisor classes, we have the following theorem (it can be confirmed directly that $\widetilde{W}(D_5^{(1)})$ does preserve effectiveness by realization as automorphisms of the family of surfaces).

Theorem 3.2 *The group of Cremona isometries of the surface X is $\widetilde{W}(D_5^{(1)})$.*

Corollary 3.3 *The mapping φ^2 acts on the root basis as*

$$\varphi^2 : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto$$
$$(\alpha_0 + K, \alpha_1 + K, \alpha_2 - K, \alpha_3 - K, \alpha_4 + K, \alpha_5 + K)$$

where $K = -K_X = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$ corresponds to the canonical central element and φ is written by generators as

$$\varphi = \sigma_{54} \circ \sigma_{10} \circ w_5 \circ w_4 \circ w_1 \circ w_0 \circ w_2 \circ w_3 \circ w_2.$$

Remark. Since we have defined the composition of mappings as in Sect. 2, the composition of actions on the Picard group should like a changes of basis, i.e. for $\varphi : (x, y) \mapsto (2x, y)$ and $\psi : (x, y) \mapsto (x + y, y)$, we have $\psi \circ \varphi : (x, y) \mapsto (2x + 2y, y)$.

3.3 Realization of Cremona isometries

Each Cremona isometry of Th.3.2 can be realized as birational mapping on $\mathbb{P}^1 \times \mathbb{P}^1$ which is lifted to an automorphism of the family of surfaces X (cf. [11, 12]).

The birational actions of generators are as follows: $(x, y, a_0, a_1, a_2, a_3, c, d)$ is mapped

to

$$\left(x, y\sqrt{\frac{a_3}{a_1 d}};\ a_0, \sqrt{\frac{a_1 a_3}{d}}, a_2, \sqrt{a_1 a_3 d}, c, \frac{a_3}{a_1}\right) \qquad \text{by } w_0$$

$$\left(x, y\sqrt{\frac{a_1}{a_3 d}};\ a_0, \sqrt{a_1 a_3 d}, a_2, \sqrt{\frac{a_1 a_3}{d}}, c, \frac{a_1}{a_3}\right) \qquad \text{by } w_1$$

$$\left(\frac{x(a_1 y+1)}{(y+a_3 d)}\sqrt{\frac{a_3 d}{a_1}}, \frac{y}{\sqrt{a_1 a_3 d}};\ a_0\sqrt{a_1 a_3 d}, \sqrt{\frac{a_1}{a_3 d}}, a_2\sqrt{a_1 a_3 d}, \sqrt{\frac{a_3}{a_1 d}}, c, \frac{1}{a_1 a_3}\right) \quad \text{by } w_2$$

$$\left(\frac{x}{\sqrt{a_0 a_2 c}}, \frac{y(x+a_2 c)}{(a_0 x+1)}\sqrt{\frac{a_0}{a_2 c}};\ \sqrt{\frac{a_0}{a_2 c}}, a_1\sqrt{a_0 a_2 c}, \sqrt{\frac{a_2}{a_0 c}}, a_3\sqrt{a_0 a_2 c}, \frac{1}{a_0 a_2}, d\right) \quad \text{by } w_3$$

$$\left(x\sqrt{\frac{a_2}{a_0 c}}, y;\ \sqrt{\frac{a_0 a_2}{c}}, a_1, \sqrt{a_0 a_2 c}, a_3, \frac{a_2}{a_0}, d\right) \qquad \text{by } w_4$$

$$\left(x\sqrt{\frac{a_0}{a_2 c}}, y;\ \sqrt{a_0 a_2 c}, a_1, \sqrt{\frac{a_0 a_2}{c}}, a_3, \frac{a_0}{a_2}, d\right) \qquad \text{by } w_5$$

$$\left(\frac{1}{x}, \frac{y}{d};\ \frac{1}{a_0}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{c}, \frac{1}{d}\right) \qquad \text{by } \sigma_{10}$$

$$\left(\frac{y}{d}, \frac{x}{c};\ \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{a_0}, \frac{1}{d}, \frac{1}{c}\right) \qquad \text{by } \sigma_{543210}\ .$$

Remark. If one would prefer to relate these transformation to Sakai's paper [11] explicitly, it is sufficient to put f, g and $b_1, b_2, \cdots, b_8$ as $f = x, g = y$ and $b_1 = -a_1, b_2 = -da_3, b_3 = -1/a_1, b_4 = -d/a_3, b_5 = -1/a_0, b_6 = -c/a_2, b_7 = -a_0, b_8 = -ca_2$. (This coincidence is up to the coefficients of f and g in each birational transformation, because the normalization is different in Sakai's paper. While there are 8 parameters in that paper, there are only 6 parameters here. If we consider surfaces whose Picard group and effective classes are the same as those of the surfaces X and normalize by automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$, then only "6" parameters remain.)

3.4 Relation to the q-Painlevé VI equation

We define the mapping ψ as $\psi = \sigma_{10} \circ \sigma_{543210} \circ w_3 \circ w_5 \circ w_4 \circ w_3$, then ψ is expressed as

$$\psi : (x, y) \ \mapsto \ (\overline{x}, \overline{y}),$$

$$\begin{cases} \overline{x} = \dfrac{d(1 + a_0 x)(c + a_2 x)}{y(x + a_0)(x + ca_2)} \\ \overline{y} = x \end{cases} \tag{7}$$

with the change of parameters:

$$(\overline{a_0}, \overline{a_1}, \overline{a_2}, \overline{q}, \overline{c}, \overline{d}) = (\frac{a_0}{q}, qa_1, \frac{a_2}{q}, q, d, c).$$

The mapping ψ^2 is the q-P_{VI} equation. Actually it acts on the root basis as

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_0, \alpha_1, \alpha_2 + K, \alpha_3 - K, \alpha_4, \alpha_5).$$

Remark. If one would prefer to relate these transformation to Sakai's qP_{VI} [11], it is sufficient to put $f = x, g = y$ and $b_1 = -ta_1, b_2 = -tda_3, b_3 = -t/a_1, b_4 = -td/a_3, b_5 = -t/a_0, b_6 = -tc/a_2, b_7 = -ta_0, b_8 = -tca_2$ and $\overline{t} = qt$ and $q' = q^2$ (the normalization is changed by introducing the new variable t).

8

Proposition 3.4 *The mapping φ^2 can be decomposed into conjugate mappings of q-$P_{VI}(= \psi^2)$ as*

$$\varphi^2 = (\sigma_{543210} \circ w_2 \circ q\text{-}P_{VI} \circ w_2 \circ \sigma_{543210}) \circ (w_2 \circ q\text{-}P_{VI} \circ w_2).$$

4   The group of commutative elements

In this section we investigate the subgroup of $\widetilde{W}(D_5^{(1)})$ whose elements commute with φ^2 (we denote it as W_C). We prove the following theorem.

Theorem 4.1
$$W_C \simeq \mathbb{Z} \times \widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$$

holds, where the meaning of each symbol is:
i) $\mathbb{Z}$ *is* $\varphi^{2\mathbb{Z}}$;
ii) $\widetilde{W}(A_3^{(1)})$ *is the extended Weyl group defined by the root basis*

$$(\beta_0, \beta_1, \beta_2, \beta_3) := (\alpha_3 + \alpha_5, \ \alpha_1 + \alpha_2, \ \alpha_3 + \alpha_4, \ \alpha_0 + \alpha_2)$$

and the automorphisms of Dynkin diagram $\sigma_{\beta(1032)}$ and $\sigma_{\beta(3210)}$; It coincides with the original $W(A_3^{(1)})$ except for the extension;
iii) $\widetilde{W}(A_1^{(1)})$ *is the extended Weyl group defined by the root basis*

$$(\gamma_0, \gamma_1) := (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, \ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)$$

and the automorphisms of Dynkin diagram $\sigma_{\gamma(10)}$.

Notice that the subset of the lattice Q (5) whose elements are preserved by φ^2 is

$$Q_B := \{\sum_{i=0}^{5} a_i \alpha_i \in Q; \ a_0 + a_1 - a_2 - a_3 + a_4 + a_5 = 0\}.$$

The lattice Q_B can be described as

$$\begin{aligned} \alpha \in Q_B \ &= \ \mathrm{Span}_{\mathbb{Z}}(\beta_0, \beta_1, \beta_2, \beta_3, \gamma_0) \\ &= \ \mathrm{Span}_{\mathbb{Z}}(\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1). \end{aligned} \quad (8)$$

Lemma 4.2 *The group W_C preserves the lattice Q_B.*

Proof. Let $w_c \in W_C$ and let $q \in Q_B$. Since $\varphi^2 \circ w_c(q) = w_c \circ \varphi^2(q) = w_c(q)$, $w_c(q)$ is preserved by φ^2 and therefore $w_c(q) \in Q_B$. □

Lemma 4.3 *The group whose actions preserve the lattice Q_B and the intersection form is $\pm\widetilde{W}(A_3^{(1)}) \times \pm\widetilde{W}(A_1^{(1)})$.*

Proof. First we consider $w(\beta_i)$. $w(\beta_i)$ is written as

$$w(\beta_i) = b_{i,0}\beta_0 + b_{i,1}\beta_1 + b_{i,2}\beta_2 + b_{i,3}\beta_3 + c_{i,0}\gamma_0,$$

where $b_{i,j}, c_{i,j} \in \mathbb{Z}$. From the equation

$$
\begin{aligned}
-2 &= \beta_i \cdot \beta_i = w(\beta_i) \cdot w(\beta_i) \\
&= -(b_{i,0} - b_{i,1})^2 - (b_{i,1} - b_{i,2})^2 - (b_{i,2} - b_{i,3})^2 - (b_{i,3} - b_{i,0})^2 - 2(c_{i,0})^2,
\end{aligned}
$$

we have $b_{i,0} = b_{i,1} = b_{i,2} = b_{i,3}$ or $c_{i,0} = 0$. In the former case we have that

$$1 = w(\beta_i) \cdot w(\beta_{i+1}) = -2c_{i,0}c_{i+1,0},$$

which is a contradiction. Hence w preserves the lattice $\mathbb{Z}\beta_0 + \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 + \mathbb{Z}\beta_3$ and therefore the action of w on this lattice coincides with an element of the extended affine Weyl group $\pm\widetilde{W}(A_3^{(1)})$ (See [4] §5.10).

Next we consider $w(\gamma_0)$ and $w(\gamma_1)$. $w(\gamma_i)$ is written as

$$w(\gamma_i) = b_{i,0}\beta_0 + b_{i,1}\beta_1 + b_{i,2}\beta_2 + b_{i,3}\beta_3 + c_{i,0}\gamma_0$$

and we have $b_{i,0} = b_{i,1} = b_{i,2} = b_{i,3}$ or $c_{i,0} = 0$ for each i. Since the rank of $w(Q_B)$ equals that of Q_B, $c_{i,0}$ can not be zero. Hence we have $w(\gamma_i) \in \mathbb{Z}K \pm \gamma_0 \subset \mathbb{Z}\gamma_0 + \mathbb{Z}\gamma_1$ and therefore the action of w on the lattice $\mathbb{Z}\gamma_0 + \mathbb{Z}\gamma_1$ coincides with an element of the extended affine Weyl group $\pm\widetilde{W}(A_1^{(1)})$. $\qquad\square$

Now we are ready to prove the theorem.

Proof. By Lemma 4.2 and Lemma 4.3 it follows that the action of $W_C \subset \widetilde{W}(D_5^{(1)})$ on Q_B is given by $\widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$. One of the extensions of this action onto Q is given in appendix B.

We investigate the variety of extensions. Suppose the actions of $s_1, s_2 \in W_C$ on the lattice Q_B are the same. We write $s_1 \circ s_2^{-1}(x)$ as $\overline{x}$. Since $\alpha_0, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_0$ are linearly independent, $\overline{\alpha_0}$ can be written as

$$a_0\alpha_0 + b_0\beta_0 + b_1\beta_1 + b_2\beta_2 + b_3\beta_3 + c_0\gamma_0.$$

Using the relations $\alpha_0 \cdot \beta_i = \overline{\alpha_0} \cdot \overline{\beta_i} = \overline{\alpha_0} \cdot \beta_i$ and $\alpha_0 \cdot \gamma_0 = \overline{\alpha_0} \cdot \overline{\gamma_0} = \overline{\alpha_0} \cdot \gamma_0$, we have

$$
\begin{cases}
-2b_0 + b_1 + b_3 &= 0 \\
a_0 + b_0 - 2b_1 + b_2 &= 1 \\
b_1 - 2b_2 + b_3 &= 0 \\
-a_0 + b_0 + b_2 - 2b_3 &= -1 \\
-a_0 - 2c_0 &= -1.
\end{cases}
$$

and hence

$$
\begin{aligned}
\overline{\alpha_0} &= \alpha_0 + \mathbb{Z}K + \mathbb{Z}(2\alpha_0 + \beta_0 + 2\beta_1 + \beta_2 - \gamma_0) \\
&= \alpha_0 + \mathbb{Z}K + \mathbb{Z}(\alpha_2 + \alpha_3).
\end{aligned}
$$

Moreover from $(\overline{\alpha_0})^2 = (\alpha_0)^2 = -2$, we have

$$\overline{\alpha_0} = \alpha_0 + \mathbb{Z}K \text{ or } \alpha_0 + \alpha_2 + \alpha_3 + \mathbb{Z}K. \tag{9}$$

Since $\overline{\beta_i} = \beta_i$ and $\overline{\gamma_0} = \gamma_0$, we have: if $\overline{\alpha_0} = \alpha_0 + zK$, $(z \in \mathbb{Z})$, then

$$(\overline{\alpha_0}, \overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5})$$
$$= (\alpha_0 + zK, \alpha_1 + zK, \alpha_2 - zK, al_3 - zK, \alpha_4 + zK, \alpha_5 + zK)$$

and if $\overline{\alpha_0} = \alpha_0 + \alpha_2 + \alpha_3 + zK$, $(z \in \mathbb{Z})$, then

$$(\overline{\alpha_0}, \overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5})$$
$$= (\alpha_0 + \alpha_2 + \alpha_3 + zK, \alpha_1 + \alpha_2 + \alpha_3 + zK, -\alpha_3 - zK,$$
$$-\alpha_2 - zK, \alpha_2 + \alpha_3 + \alpha_4 + zK, \alpha_2 + \alpha_3 + \alpha_5 + zK).$$

In the former case $s_1 \circ s_2^{-1} = \varphi^2 = (r_1 \circ r_0)^z$ and in the later case $s_1 \circ s_2^{-1} = \varphi^2 = r_0 \circ (r_1 \circ r_0)^z$. Since r_0 does not commute with φ^2 $(r_0 \circ \varphi^2 = \varphi^{-2} \circ r_0)$, the action of W_B is extended by $\varphi^{2\mathbb{Z}}$. $\square$

A   Embedding $\widetilde{W}(A_1^{(1)} \times A_3^{(1)})$ of KNY into $\widetilde{W}(D_5^{(1)})$

By studying the actions of the elements of original Weyl group $\widetilde{W}(A_1^{(1)} \times A_3^{(1)})$ on the root basis, we can realize its embedding into the $\widetilde{W}(D_5^{(1)})$ as follows

$$
\begin{aligned}
r_0 &= w_2 \circ w_3 \circ w_2 \\
r_1 &= w_0 \circ w_1 \circ w_4 \circ w_5 \circ w_2 \circ w_3 \circ w_2 \circ w_0 \circ w_1 \circ w_4 \circ w_5 \\
\pi &= w_0 \circ w_1 \circ w_4 \circ w_5 \circ \sigma_{10} \circ \sigma_{54} \\
s_0 &= w_5 \circ w_3 \circ w_5 \\
s_1 &= w_1 \circ w_2 \circ w_1 \\
s_2 &= w_3 \circ w_4 \circ w_3 \\
s_3 &= w_0 \circ w_2 \circ w_0 \\
\rho &= \sigma_{543210} \circ \sigma_{10}.
\end{aligned}
$$

B   Embedding the group W_C into $\widetilde{W}(D_5^{(1)})$

As mentioned in Section 4 the actions of $\widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$ on the lattice Q_B are not uniquely expanded to onto the lattice Q. The variety of extensions is $\varphi^{2\mathbb{Z}} = (r_1 \circ r_0)^{\mathbb{Z}}$. Here, we give one of the embeddings of $\widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$ into $\widetilde{W}(D_5^{(1)})$.

For $\beta = \beta_i$ or γ_i we define the action on the lattice Q as

$$w_\beta(\alpha) = \alpha - 2\frac{\beta \cdot \alpha}{\beta \cdot \beta}\beta,$$

where $\alpha \in Q$. In order to find its description in terms of the fundamental elements of $\widetilde{W}(D_5^{(1)})$, we use the relation

$$w_{w_\alpha(\beta)} = w_\alpha \circ w_\beta \circ w_\alpha,$$

where α and β are real roots.

$$
\begin{aligned}
w_{\beta_0} &= w_{\alpha_3+\alpha_5} = w_3 \circ w_5 \circ w_3 \\
w_{\beta_1} &= w_{\alpha_1+\alpha_2} = w_1 \circ w_2 \circ w_1 \\
w_{\beta_2} &= w_{\alpha_3+\alpha_4} = w_3 \circ w_4 \circ w_3 \\
w_{\beta_3} &= w_{\alpha_0+\alpha_2} = w_0 \circ w_2 \circ w_0 \\
w_{\gamma_0} &= w_{\alpha_0+\alpha_1+\alpha_2+\alpha_3} = w_3 \circ w_0 \circ w_2 \circ w_1 \circ w_2 \circ w_0 \circ w_3 \\
w_{\gamma_1} &= w_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} = w_2 \circ w_5 \circ w_3 \circ w_4 \circ w_3 \circ w_5 \circ w_2
\end{aligned}
$$

Finally, we define the action of automorphisms of the Dynkin diagram of $\widetilde{W}(A_3^{(1)}) \times \widetilde{W}(A_1^{(1)})$: $\sigma_{\beta(1032)}, \sigma_{\beta(1032)}, \sigma_{\gamma(10)}$ as $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is mapped to

$$
\begin{aligned}
\text{by } \sigma_{\beta(1032)}: \quad &(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, -\alpha_0 - \alpha_1 - \alpha_2, \\
&-\alpha_3 - \alpha_4 - \alpha_5, \alpha_0 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \\
\text{by } \sigma_{\beta(3210)}: \quad &(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, -\alpha_0 - \alpha_1 - \alpha_2, \\
&-\alpha_3 - \alpha_4 - \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_0 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \\
\text{by } \sigma_{\gamma(10)}: \quad &(\alpha_0 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, -\alpha_3 - \alpha_4 - \alpha_5, \\
&-\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5).
\end{aligned}
$$

Hence, these are written in termes of the fundamental elements of $\widetilde{W}(D_5^{(1)})$ as:

$$
\begin{aligned}
\sigma_{\beta(1032)} &= \sigma_{543210} \circ w_5 \circ w_4 \circ w_1 \circ w_0 \circ w_2 \circ w_3 \circ w_2 \\
\sigma_{\beta(3210)} &= \sigma_{543210} \circ \sigma_{54} \circ \sigma_{10} \circ w_5 \circ w_4 \circ w_1 \circ w_0 \circ w_2 \circ w_3 \circ w_2 \\
\sigma_{\gamma(10)} &= \varphi = \sigma_{54} \circ \sigma_{10} \circ w_5 \circ w_4 \circ w_1 \circ w_0 \circ w_2 \circ w_3 \circ w_2.
\end{aligned}
$$

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